

# A seventeenth-order polylogarithm ladder<sup>a)</sup>

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**Abstract** Cohen, Lewin and Zagier found four ladders that entail the polylogarithms  $\text{Li}_n(\alpha_1^{-k}) := \sum_{r>0} \alpha_1^{-kr}/r^n$  at order  $n = 16$ , with indices  $k \leq 360$ , and  $\alpha_1$  being the smallest known Salem number, i.e. the larger real root of Lehmer's celebrated polynomial  $\alpha^{10} + \alpha^9 - \alpha^7 - \alpha^6 - \alpha^5 - \alpha^4 - \alpha^3 + \alpha + 1$ , with the smallest known non-trivial Mahler measure. By adjoining the index  $k = 630$ , we generate a fifth ladder at order 16 and a ladder at order 17 that we presume to be unique. This empirical integer relation, between elements of  $\{\text{Li}_{17}(\alpha_1^{-k}) \mid 0 \leq k \leq 630\}$  and  $\{\pi^{2j}(\log \alpha_1)^{17-2j} \mid 0 \leq j \leq 8\}$ , entails 125 constants, multiplied by integers with nearly 300 digits. It has been checked to more than 59,000 decimal digits. Among the ladders that we found in other number fields, the longest has order 13 and index 294. It is based on  $\alpha^{10} - \alpha^6 - \alpha^5 - \alpha^4 + 1$ , which gives the sole Salem number  $\alpha < 1.3$  with degree  $d < 12$  for which  $\alpha^{1/2} + \alpha^{-1/2}$  fails to be the largest eigenvalue of the adjacency matrix of a graph.

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<sup>a)</sup> This work was supported by the Director, Office of Computational and Technology Research, Division of Mathematical, Information, and Computational Sciences of the U.S. Department of Energy, under contract number DE-AC03-76SF00098.

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# 1 Introduction

The findings reported here arose from the discovery by the second author that

$$\alpha^{630} - 1 = \frac{(\alpha^{315} - 1)(\alpha^{210} - 1)(\alpha^{126} - 1)^2(\alpha^{90} - 1)(\alpha^3 - 1)^3(\alpha^2 - 1)^5(\alpha - 1)^3}{(\alpha^{35} - 1)(\alpha^{15} - 1)^2(\alpha^{14} - 1)^2(\alpha^5 - 1)^6\alpha^{68}} \quad (1)$$

where  $\alpha$  is one of the 10 algebraic integers of Lehmer's remarkable number field [12]

$$\alpha^{10} + \alpha^9 - \alpha^7 - \alpha^6 - \alpha^5 - \alpha^4 - \alpha^3 + \alpha + 1 = 0 \quad (2)$$

Once found, the cyclotomic relation (1) was proven by (oft) repeated substitution for  $\alpha^{10}$ . It led us to believe that a valid ladder of polylogarithms exists at order  $n = 17$ , contrary to a suggestion in [18]. Indeed, we were able to adjoin the index  $k = 630$  to those with  $k \leq 360$ , found by Henri Cohen, Leonard Lewin and Don Zagier [8], and obtain

$$N(n) = 77 - \lfloor 9n/2 \rfloor \quad (3)$$

ladders at orders  $n = 2 \dots 17$ . In particular, at  $n = 17$ , we found 125 non-zero integers<sup>1</sup>  $a$ ,  $b_j$ ,  $c_k$ , with less than 300 digits, such that an empirical relation

$$a \zeta(17) = \sum_{j=0}^8 b_j \pi^{2j} (\log \alpha_1)^{17-2j} + \sum_{k \in D(\mathcal{S})} c_k \text{Li}_{17}(\alpha_1^{-k}) \quad (4)$$

holds to more than 59,000 decimal digits, where

$$\alpha_1 = 1.176280818259917506544070338474035050693415806564 \dots \quad (5)$$

is the larger real root of (2), and the 115 indices  $k$  in  $\text{Li}_n(\alpha_1^{-k}) := \sum_{r>0} \alpha_1^{-kr}/r^n$  are drawn from the set,  $D(\mathcal{S})$ , of positive integers that divide at least one element of

$$\mathcal{S} := \{29, 47, 50, 52, 56, 57, 64, 74, 75, 76, 78, 84, 86, 92, 96, 98, 108, 110, 118, 124, 130, 132, 138, 144, 154, 160, 165, 175, 182, 186, 195, 204, 212, 240, 246, 270, 286, 360, 630\} \quad (6)$$

The coefficient of  $\zeta(17)$  was partially factorized as follows

$$a = 2^7 \times 3^7 \times 5^4 \times 7 \times 11 \times 13 \times 17 \times 722063 \times 15121339 \times 379780242109750106753 \\ \times 5724771750303829791195961 \times C_{217} \quad (7)$$

where

$$C_{217} := 5203751052922114540188667952627280712081039342696719260003747081 \\ 41977100981249686783730105404186042839389917052601102889831046723208680 \\ 07066945997308654073833814804516883406394532403532415753816146816138731 \\ 90080853089 \quad (8)$$

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<sup>1</sup>See <ftp://physics.open.ac.uk/pub/physics/dbroadhu/lehmer/integers.txt>

is a 217-digit non-prime, whose factorization has not yet been obtained. The integers in (4) were obtained using less than 4,000 digits of working precision. The chance of a numerical accident is thus less than  $10^{-55000}$ .

In section 2, we review the algorithm for generating such ladders from cyclotomic relations [18]. In section 3, we describe our computational strategy, based on the PSLQ integer relation finder [3]. In section 4, we study ladders based on Salem numbers larger than (5), commenting on a connection to graph theory, observed with Gert Almkvist.

## 2 Ladder building in self-reciprocal number fields

Consider the cyclotomic polynomials,  $\Phi_k(x)$ , defined recursively by

$$x^k - 1 = \prod_{j|k} \Phi_j(x) \quad (9)$$

A real algebraic number  $x > 1$  is said to satisfy a cyclotomic relation with index  $k$  if there exist rational numbers  $\{A_j \mid 0 \leq j < k\}$  such that

$$\Phi_k(x) = x^{A_0} \prod_{j=1}^{k-1} \Phi_j^{A_j}(x) \quad (10)$$

For example, identity (1) establishes that  $\alpha_1$  – the smallest known Salem number, and also the smallest known non-trivial Mahler measure – satisfies a cyclotomic relation with index  $k = 630$ , which is 63 times larger than the degree of  $\alpha_1$ . This ratio is larger than any heretofore discovered, the previous [8] record being  $k/d = 36$ .

A Salem number is a real algebraic integer,  $\alpha > 1$ , of degree  $d = 2 + 2s$ , with  $2s > 0$  conjugates on the unit circle and the remaining conjugate,  $1/\alpha$ , inside it. For any monic polynomial,  $P$ , with integer coefficients, the Mahler measure,  $M(P)$ , is the product of the absolute values of the roots outside the unit circle. Thus a Salem number is the Mahler measure of its minimal polynomial. Derrick Lehmer [12] conjectured that there exists a constant  $c > 1$  such that  $M(P) \geq c$  for all  $M(P) > 1$ . A stronger form of this conjecture is that  $c = \alpha_1$ , the Mahler measure of (2), found by Lehmer more than 60 years ago, and still [4, 5, 16] the smallest known  $M(P) > 1$ .

In describing how to build a polylogarithmic ladder, we shall restrict attention to an algebraic number field whose defining polynomial,  $P(x) = x^d P(1/x)$ , of even degree  $d$ , is reciprocal (i.e. palindromic), as in (2). Moreover, we require at least one real root with  $x > 1$ . Hence the discussion encompasses all Salem numbers. We define cyclotomic norms [18]

$$N_k := \prod_{r=1}^d \Phi_k(x_r) \quad (11)$$

where the product is over all roots of  $P(x)$ , so that  $N_k$  is an integer. Then a necessary condition for a cyclotomic relation of index  $k$  is that every prime factor of  $N_k$  is also a factor of a norm  $N_j$  with  $j < k$ . This simplifies, considerably, the business of finding all

indices of cyclotomic relations, up to some maximum value, which we set at 1800, i.e. five times larger than the previous record  $k = 360$ , set by [8].

First, one rules out indices that fail the factorization criterion. Then, for each surviving  $k$ , one performs an integer relation search, at suitably chosen numerical precision, using the constants  $\log x$ ,  $\log \Phi_k(x)$  and a subset of  $\{\log \Phi_j(x) \mid j < k\}$  that is consistent with the requirement that

$$N_k = \prod_{0 < j < k} N_j^{A_j} \quad (12)$$

be satisfiable by rational numbers  $A_j$ . Moreover, this subset can be further – and often greatly – reduced by exploiting the cyclotomic relations with indices less than  $k$ . In fact, we obtained integer values of  $A_j$  that vanish for each  $j > d/2$  that does not divide  $k$ .

For each putative cyclotomic relation thus indicated by a numerically discovered integer relation between logarithms, one has merely to use the defining polynomial, to eliminate  $x^d$ , via a computer algebra program, hence proving (or if one is very unlucky disproving) the numerically suggested relation. This is how we discovered and proved the cyclotomic relation (1), with index  $k = 630$ , in the number field defined by Lehmer's polynomial (2). We are also strongly convinced that there is no cyclotomic relation with  $630 < k \leq 1800$  in Lehmer's number field.

Taking the log of a cyclotomic relation, with a real root  $x > 1$ , one proves the vanishing of a combination of logarithms of the form

$$\text{Li}_1(x^{-k}) - B_0 \log x + \sum_{0 < j < k} B_j \text{Li}_1(x^{-j}) = 0 \quad (13)$$

where the logarithm  $\text{Li}_1(y) := -\log(1 - y)$  is merely the  $n = 1$  case of the polylogarithm  $\text{Li}_n(y) := \sum_{r>0} y^r/r^n$ , with order  $n$ , and the rational numbers  $B_j$  are easily obtained from the rational numbers  $A_j$  in the cyclotomic relation (10).

Next, we define polylogarithmic combinations

$$L_k^{(1)}(n) := \text{Li}_n(x^{-k})/k^{n-1} + B_0(-\log x)^n/n! + \sum_{0 < j < k} B_j \text{Li}_n(x^{-j})/j^{n-1} \quad (14)$$

where the superscript indicates that they all vanish at  $n = 1$ . In general, these do not vanish at  $n = 2$ . Rather, one finds [15] that combinations of them evaluate to rational multiples of  $\zeta(2) = \pi^2/6$ .

Suppose that several  $\mathbf{Q}$ -linear combinations of the constructs (14) evaluate to rational multiples of  $\pi^2$  at  $n = 2$ , so that

$$\sum_k C_{jk}^{(2)} L_k^{(1)}(2) = D_j^{(2)} \pi^2 \quad (15)$$

with a rational matrix  $C^{(2)}$  yielding a rational vector  $D^{(2)}$ , where the superscript indicates that we have exploited empirical data at order  $n = 2$ . Then one forms a vector whose components

$$L_j^{(2)}(n) = \sum_k C_{jk}^{(2)} L_k^{(1)}(n) - D_j^{(2)} \frac{\pi^2 (-\log x)^{n-2}}{(n-2)!} \quad (16)$$

vanish at  $n = 2$ . At  $n = 3$ , one seeks  $\mathbf{Q}$ -linear combinations of (16) that evaluate, empirically, to rational multiples of  $\zeta(3)$ . However, these are not yet the constructs to carry forward to orders  $n > 3$ ; one must form combinations that vanish at  $n = 3$ . In a self-reciprocal number field, this cannot be done by a subtraction similar to that in (16), since  $\zeta(3)$  does not appear in the formula [14] for inverting the argument of a polylogarithm.

Thus the generic iteration is to form combinations

$$L_j^{(2p+1)}(n) = \sum_k C_{jk}^{(2p+1)} L_k^{(2p)}(n) \quad (17)$$

that vanish for  $n = 2p + 1$ , and then combinations

$$L_j^{(2p+2)}(n) = \sum_k C_{jk}^{(2p+2)} L_k^{(2p+1)}(n) - D_j^{(2p+2)} \frac{\pi^{2p+2} (-\log x)^{n-2p-2}}{(n-2p-2)!} \quad (18)$$

that vanish for  $n = 2p + 2$ .

The vital issue is this: how does the number of valid ladders decrease at each iteration? Don Zagier observed that the answer depends on the signature of the number field [18]. Suppose that the polynomial  $P(x)$  has  $r > 0$  real roots and  $s$  pairs of complex roots, so that the degree is  $d = r + 2s$ . (Note that  $r = 2$  and  $s > 0$  in the particular case of a Salem number.) Since we restrict attention to reciprocal polynomials, both  $r$  and  $d$  are even. (In particular, the Lehmer polynomial has  $r = 2$ ,  $s = 4$ ,  $d = 10$ .) Given  $N(2p)$  ladders that evaluate to rational multiples of  $\pi^{2p}$  at order  $n = 2p$ , one expects

$$N(2p+1) = N(2p) + 1 - d/2 \quad (19)$$

ladders that evaluate to rational multiples of  $\zeta(2p+1)$  at order  $n = 2p + 1$ , and

$$N(2p+2) = N(2p+1) - 1 - s \quad (20)$$

ladders that evaluate to rational multiples of  $\pi^{2p+2}$  at order  $n = 2p + 2$ . The  $+1$  in (19) occurs because one may include the index  $k = 0$ , corresponding to  $\zeta(2p+1)$ ; the  $-1$  in (20) because this odd zeta value may not be carried forward. The  $-d/2$  in (19) occurs because of conditions on the functionally independent real parts of polylogs of odd order, in a self-reciprocal number field; the  $-s$  in (20) because of conditions on the imaginary parts of polylogs of even order.

For a self-reciprocal number field with degree  $d = 2 + 2s$  – and hence for any Salem number – it follows that  $C$  cyclotomic relations are expected to generate

$$N(n) = C + d/2 - \lfloor (d-1)n/2 \rfloor \quad (21)$$

rational multiples of  $\zeta(n)$  at order  $n \geq 2$ .

By way of example, the self-reciprocal number field  $\alpha^2 - 3\alpha + 1 = 0$  has  $C = 4$  cyclotomic relations, with indices  $k = 1, 6, 10, 12$ . There are thus  $5 - \lfloor n/2 \rfloor$  valid ladders at orders  $n = 2 \dots 9$ . At  $n = 9$ , there is an integer relation

$$f \zeta(9) = \sum_{j=0}^4 g_j \pi^{2j} (\log \phi)^{9-2j} + \sum_{k \in D(\{10,12\})} h_k \text{Li}_9(\phi^{-2k}) \quad (22)$$

where  $\phi := (1 + \sqrt{5})/2$  is the golden ratio and  $f, g_j, h_k$  are essentially unique integers, with indices  $k$  dividing 10 or 12. In this very simple case, empirical [15] determination of

$$f = 2 \times 3^3 \times 5 \times 7^2 \times 23 \times 191 \times 2161 \quad (23)$$

requires only Euclid's algorithm, to find the rational ratio of each previously vanishing ladder to the current zeta value.

Much more impressively, Cohen, Lewin and Zagier [8] found 71 cyclotomic relations with indices  $k \leq 360$ , for the smallest known Salem number, with  $d = 10$ . Thus they obtained  $71 + 5 - 9 \times 8 = 4$  valid ladders at order  $n = 16$ , yet no relation at  $n = 17$ .

### 3 The Lehmer ladder of order 17

The tables on pages 368–370 of [15] exhaust the cyclotomic relations with indices  $k \leq 360$  in the Lehmer number field. It seemed to us peculiarly inconvenient that this tally was precisely one short of what is needed to generate a rational multiple of  $\zeta(17)$ . It was also clear how to look for “the one that got away”. We calculated the norms  $N_k$  of  $\{\Phi_k(\alpha) \mid 360 < k \leq 1800\}$  and found only one candidate with small factors, namely  $N_{630} = N_{126} = 5^6$ . It seemed likely that  $\Phi_{630}(\alpha)/\Phi_{126}(\alpha)$  was party to a cyclotomic relation. Taking logarithms and using PSLQ, we readily found the numerical relation

$$\frac{\Phi_{630}(\alpha_1)}{\Phi_{126}(\alpha_1)} = \frac{\alpha_1^{58} - \alpha_1^{55}}{(\alpha_1 - 1)^5} \quad (24)$$

which was then proven by repeated substitution for  $\alpha_1^{10}$ . It entails terms of the form  $\alpha_1^j - 1$ , where  $j$  is one of the 24 divisors of 630. These may be halved in number, as in (1), by eliminating the 12 divisors  $j \in \{6, 7, 9, 10, 18, 21, 30, 42, 45, 63, 70, 105\}$ , which are themselves [15] indices of cyclotomic relations.

Proceeding to dilogarithms, we then needed to perform only 6-dimensional searches for integer relations, between the constants  $\zeta(2)$  and  $\{L_j^{(1)}(2) \mid j = 6, 7, 8, 9, k\}$ , in the 68 cyclotomic cases with  $9 < k \leq 630$ . This resulted in the 67 dilogarithmic ladders of [15] and one new ladder, namely the integer relation

$$\begin{aligned} 0 &= \text{Li}_2(\alpha_1^{-630}) - 2 \text{Li}_2(\alpha_1^{-315}) - 3 \text{Li}_2(\alpha_1^{-210}) - 10 \text{Li}_2(\alpha_1^{-126}) - 7 \text{Li}_2(\alpha_1^{-90}) \\ &+ 18 \text{Li}_2(\alpha_1^{-35}) + 84 \text{Li}_2(\alpha_1^{-15}) + 90 \text{Li}_2(\alpha_1^{-14}) - 4 \text{Li}_2(\alpha_1^{-9}) + 339 \text{Li}_2(\alpha_1^{-8}) \\ &+ 45 \text{Li}_2(\alpha_1^{-7}) + 265 \text{Li}_2(\alpha_1^{-6}) - 273 \text{Li}_2(\alpha_1^{-5}) - 678 \text{Li}_2(\alpha_1^{-4}) - 1016 \text{Li}_2(\alpha_1^{-3}) \\ &- 744 \text{Li}_2(\alpha_1^{-2}) - 804 \text{Li}_2(\alpha_1^{-1}) - 22050 (\log \alpha_1)^2 + 2003 \zeta(2) \end{aligned} \quad (25)$$

whose index,  $k = 630$ , exceeds anything found previously. We remark that the coefficients of  $\{\text{Li}_2(\alpha_1^{-j}) \mid 9 < j < 630\}$  are determined by (1) and that the empirical coefficient of  $\zeta(2)$  is a 4-digit prime, namely 2003.

At this juncture, we were faced by a computational dilemma: how should one process empirical rational data, at orders  $n < 17$ , so as fastest to determine the final order-17

ladder? There are two radically opposed strategies: one systematic, though numerically intensive; the other interventionist, though requiring less numerical precision. In the first approach, one takes no heed of the explosion of primes, such as 2003 at  $n = 2$ . Rather, one adopts the simplest procedure of eliminating the predicted number of indices from the lowest currently available, as in the case above with  $n = 2$ , where the indices  $k = 6, 7, 8, 9$  were eliminated in passing from 72 cyclotomic relations to 68 dilogarithmic relations. This already differs from the choice adopted by Lewin in [15], who chose to eliminate the indices  $k = 7, 8, 9, 10$ , leaving  $k = 6$  as a survivor. The latter choice might, at some intermediate stage, produce integers considerably smaller than those in our method, yet it is difficult to automate an objective criterion that will efficiently limit the growth of scheme-dependent integers at all orders  $n < 17$ . Since we envisage a unique valid ladder at order 17, the choice of strategy should not affect the final result. Rather, it affects the working precision that is required.

Happily, the choice is not crucial, since state-of-the-art implementation [3] of the PSLQ algorithm enables a 6-dimensional search in seconds, when the relation involves integers with less than 600 digits. Thus we were able to experiment at lower orders. The rule of thumb which emerged is this: sub-optimal intermediate integers, produced by systematic elimination of the lowest indices currently available, rarely exceed the squares of those that might be achieved by laborious optimization.

This suggested that systematic elimination would be likely to get us fastest to the ultimate goal, without need for any tuning. It transpired that the task was indeed as easy as we had supposed, since the integer  $a$  in (7) has merely 288 digits, and all the other integers in (4) have less than 300 digits. At no stage did we encounter, in our systematic approach, an integer with more than 600 digits, consistent with the rule of thumb. Thus, in searches with merely 6 constants, we needed less than 4,000-digit working precision, which placed no significant burden on MPFUN [1, 2] or PSLQ [3, 10].

We remark that on a 433 MHz DecAlpha machine it took 139 seconds to compute the constants  $\{\text{Li}_{17}(\alpha_1^{-k}) \mid k \in D(\mathcal{S})\}$  to 4,000 digits. It then took merely 9 seconds for PSLQ to find an integer relation between  $\zeta(17)$  and the 5 valid ladders that had survived from the previous iteration. Unravelling all the iterations, we then expressed the final – and presumably unique – result in the form (4). Of the 117 divisors of (6), two are not entailed by cyclotomic relations, namely  $k = 51$  and  $k = 53$ . Among the 115 non-zero integers  $c_k$ , the cyclotomic input guarantees the triviality of 19 ratios, namely

$$\begin{aligned} \frac{c_k}{c_{2k}} &= -2^{n-1}, \quad k \in \{41, 43, 49, 59, 69, 77, 91, 93, 102, 106, 123, 135, 143, 180, 315\}; \\ \frac{c_k}{c_{3k}} &= -3^{n-1}, \quad k \in \{68, 210\}; \quad \frac{c_{82}}{c_{246}} = -2 \times 3^{n-1}; \quad \frac{c_{126}}{c_{630}} = -2 \times 5^{n-1} \end{aligned} \quad (26)$$

where, for example, the final ratio results from (24) and in the particular case of (4) the order is  $n = 17$ . Moreover, the logarithms of (4) may be removed by replacing  $\text{Li}_n(y)$  by the Rogers-type polylogarithm [15]

$$L_n(y) := \sum_{r=1}^n \left(1 - \frac{\delta_{r,1}}{n}\right) \frac{(-\log |y|)^{n-r}}{(n-r)!} \text{Li}_r(y) \quad (27)$$

which slightly modifies Kummer's [14]  $\Lambda_n(-y) := \int_0^y (\log |x|)^{n-1} dx / (1-x)$ , by an inessential normalization factor, and by a Kronecker delta term, at  $r = 1$ . The latter completes the process of removal of logs from functional equations, almost achieved by Kummer. Thus the core of relation (4) is specified by the coefficient (7) of  $\zeta(17)$  and 96 integers  $c_k$ , from which the full set of 115 is trivially generated. Analyzing these 97 integers in pairs, we found no pair with a common prime factor greater than 1973, which divides both  $c_1$  and  $c_{57}$ . From this, one sees that brute-force application of PSLQ would require nearly 30,000 digits of precision to reconstruct the relation, without benefit of further theory. In fact, 4,000 digits were more than enough to ascend the ladder (19,20), rung by rung.

We presume that the coefficient (7) of  $\zeta(17)$  is essentially unique, making it a remarkable integer in the theory of polylogarithms. Its 217-digit factor (8) is certainly composite, yet has so far resisted factorization<sup>2</sup> by Richard Crandall and his colleagues Karl Dilcher, Richard McIntosh and Alan Powell, whose large-integer code [9] efficiently implements an elliptic curve method [6]. Its presence makes it very unlikely that one could discover the final integer relation (4) with less than 1,000 digits of working precision, however hard one tried to emulate the feats of [8], by interventions that limit the growth of scheme-dependent integers at lower orders. By contrast, the ladder given in [8], at order  $n = 16$ , involved integers with no more than 71 digits, which we shall shortly reduce by 11 digits. Nonetheless, we are left speechless with admiration for the achievement of Henri Cohen in attaining order  $n = 16$  with only 305-digit precision, from Pari.

To check (4), we computed more than 59,000 digits of  $\{\text{Li}_{17}(\alpha_1^{-k}) \mid k \in D(\mathcal{S})\}$ , using

$$\frac{\text{Li}_n(\alpha_1^{-k})}{k^n} = \sum_{k|j} \frac{\alpha_1^{-j}}{j^n} \quad (28)$$

which conveniently reduces the 114 cases with  $k > 1$  to subsets of the additions for  $k = 1$ . To compute  $\zeta(17)$ , we set  $p = 4$  in the identity

$$\begin{aligned} p \zeta(4p+1) &= \frac{1}{\pi} \sum_{n=0}^{2p+1} (-1)^n \left(n - \frac{1}{2}\right) \zeta(2n) \zeta(4p+2-2n) \\ &\quad - 2 \sum_{n>0} \frac{n^{-4p-1}}{\exp(2\pi n) - 1} \left( p + \frac{\pi n}{1 - \exp(-2\pi n)} \right) \end{aligned} \quad (29)$$

which corrects the upper limit of the first sum and the sign of the second term of the second summand in Proposition 2 of [8]. The corrected companion identity simplifies to

$$\zeta(4p-1) = -\frac{1}{\pi} \sum_{n=0}^{2p} (-1)^n \zeta(2n) \zeta(4p-2n) - 2 \sum_{n>0} \frac{n^{-4p+1}}{\exp(2\pi n) - 1} \quad (30)$$

At  $p = 0$ , one finds that (29) evaluates to  $1/4$  and (30) to  $-1/12$ , as expected.

Setting  $n = 17$  in (28), we obtained the 115 polylogarithmic constants to a precision of  $3 \times 2^{16}$  binary digits. Substituting these in (4), with the integers found by PSLQ at

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<sup>2</sup>An ability to factorize the product of a pair of 100 digit primes might undermine cryptography.



less than 4,000-digit precision, we reproduced the first 59,157 decimal places of the value of  $\zeta(17)$  from (29). The chance of a spurious result is thus less than  $10^{-55000}$ .

Thanks to PSLQ, we are able to scrutinize an inference in [8], concerning the role of 3617, the numerator of the Bernoulli number  $B_{16}$ , and hence of  $\zeta(16)/\pi^{16}$ . Appendix A of [8] gives one of four empirical integer relations found at order  $n = 16$ , with indices  $k \leq 360$ . It is of a form

$$\sum_{k \in D(\mathcal{S})} a_k L_{16}(\alpha_1^{-k}) = a_0 \frac{L_{16}(1)}{3617} \quad (31)$$

On the left, one encounters in [8] 111 non-zero integers, with up to 71 digits, and on the right  $a_0$  has 75 digits. This led the authors of [8] to infer that ladders do not pick up non-trivial numerators of Bernoulli numbers, since the relation appears more natural when written in terms of  $\pi^{16}$ . However, such evidence was adduced without knowledge of the cyclotomic index  $k = 630$ .

Our first remark is that no firm conclusion may be drawn from Appendix A of [8], since it contains integers with 11 digits more than is necessary. We emphasize that this does *not* indicate a failure of Pari's LLL [13] algorithm in [8]. Rather, it shows that a new analysis is required by (1). Using PSLQ, we obtained 5 independent integer relations of the symmetrical form

$$\sum_{k \in D(\mathcal{S})} s_k L_{16}(\alpha_1^{-k}) = s_0 L_{16}(1) \quad (32)$$

with integers  $s_k$  having no more than 62 digits and Euclidean norms in the narrow range

$$2.89 \times 10^{61} < \left( s_0^2 + \sum_{k>0} s_k^2 \right)^{1/2} < 3.79 \times 10^{61} \quad (33)$$

We also obtained 5 asymmetrical relations of the form (31), with norms

$$2.31 \times 10^{61} < \left( a_0^2 + \sum_{k>0} a_k^2 \right)^{1/2} < 4.37 \times 10^{61} \quad (34)$$

Comparison of (33) with (34) leaves the issue moot. Moreover, these ranges were unchanged by application of the LLL algorithm. An indication of the slight superiority of the asymmetrical form (31) is obtained by asking LLL to reduce the restricted norms that omit the integer  $a_0$ . This produced 5 asymmetrical relations with

$$0.25 \times 10^{61} < \left( \sum_{k>0} a_k^2 \right)^{1/2} < 0.78 \times 10^{61} \quad (35)$$

The integer relation with the smallest restricted norm is presented in Table 1, whose 115 integers have no more than 60 digits and yield the 64-digit integer

$$\begin{aligned} a_0 = & 2^3 \times 11 \times 1770708910425291120521033962427 \times \\ & 23216857398851664164043691705297 \end{aligned} \quad (36)$$

with factors found after running [9] for a total of 466 CPUhours, on a cluster of 8 machines.

As further indication that powers of  $\pi^2$ , rather than even zeta values, are the constants favoured by polylogarithm ladders, we remark that  $b_8$  in (4) is not divisible by 3617, and that  $b_6$  is not divisible by 691, the numerator of  $\zeta(12)/\pi^{12}$ . Further information on the role of Bernoulli numerators in polylogarithm ladders could be obtained if one found a base in which there is a unique valid ladder of order  $n = 12$  or order  $n = 16$ . We hold out no lively hope for the latter. In search of the former, we turn to larger Salem numbers.

## 4 Ladders from larger Salem numbers

There are 47 known [16] Salem numbers less than 1.3. Of these, 45 exhaust the possibilities with  $\alpha < 1.3$  and degree  $d < 42$ . Of these, merely 6 have degree  $d < 12$ . Of these 6, we noted that all but one solve equations of the very simple form

$$x^{4+m} = \frac{Q(1/x)}{Q(x)} \quad (37)$$

with  $m > 0$  and

$$Q(x) := x^3 - x - 1 \quad (38)$$

The case  $m = 1$  gives Lehmer's number field. The minimal polynomials of the first five Salem numbers in this family are

$$P_1(\alpha) = \alpha^{10} + \alpha^9 - \alpha^7 - \alpha^6 - \alpha^5 - \alpha^4 - \alpha^3 + \alpha + 1 \quad (39)$$

$$P_2(\alpha) = \alpha^{10} - \alpha^7 - \alpha^5 - \alpha^3 + 1 \quad (40)$$

$$P_3(\alpha) = \alpha^{10} - \alpha^8 - \alpha^5 - \alpha^2 + 1 \quad (41)$$

$$P_4(\alpha) = \alpha^8 - \alpha^5 - \alpha^4 - \alpha^3 + 1 \quad (42)$$

$$P_5(\alpha) = \alpha^{10} - \alpha^8 - \alpha^7 + \alpha^5 - \alpha^3 - \alpha^2 + 1 \quad (43)$$

with approximate numerical roots – and hence Mahler measures – given by

$$\alpha_1 = 1.1762808182599175065440703384 \dots \quad (44)$$

$$\alpha_2 = 1.2303914344072247027901779389 \dots \quad (45)$$

$$\alpha_3 = 1.2612309611371388519466715030 \dots \quad (46)$$

$$\alpha_4 = 1.2806381562677575967019025327 \dots \quad (47)$$

$$\alpha_5 = 1.2934859531254541065199098837 \dots \quad (48)$$

As  $m \rightarrow \infty$ , one obtains the real root of (38), namely

$$\begin{aligned} \alpha_\infty &= \left( \frac{1 + \sqrt{23/27}}{2} \right)^{1/3} + \left( \frac{1 - \sqrt{23/27}}{2} \right)^{1/3} \\ &= 1.3247179572447460259609088544 \dots \end{aligned} \quad (49)$$

which is the smallest algebraic integer,  $\alpha > 1$ , with conjugates that all lie inside the unit circle, i.e. the smallest Pisot–Vijayaraghavan (PV) number.

We were alerted to the existence of this family of Salem numbers by graph theory. Gert Almkvist told the second author of a fascinating remark on page 247 of the book [11] by Frederick Goodman, Pierre de la Harpe and Vaughan Jones, on Coxeter graphs. There they are concerned with the smallest possible value, greater than 2, for the largest eigenvalue,  $\lambda_{\max}$ , in the spectrum of a graph. They classify all finite connected graphs with  $\phi^{1/2} + \phi^{-1/2} \geq \lambda_{\max} > 2$  where  $\phi := (1 + \sqrt{5})/2$  is the golden ratio, and hence prove that the smallest possible  $\lambda_{\max} > 2$  is obtained from the tree graph  $T_{2,3,7}$ , formed when three straight lines, with 2, 3, and 7 vertices, are joined by identifying three univalent vertices. The characteristic polynomial of its adjacency matrix is

$$P_{2,3,7}^T(\lambda) = \lambda^{10} - 9\lambda^8 + 27\lambda^6 - 31\lambda^4 + 12\lambda^2 - 1 \quad (50)$$

which is, intriguingly, related to Lehmer’s polynomial (2). The relation – which is not correctly stated in [11] – is

$$\alpha^5 P_{2,3,7}^T(\alpha^{1/2} + \alpha^{-1/2}) = \alpha^{10} + \alpha^9 - \alpha^7 - \alpha^6 - \alpha^5 - \alpha^4 - \alpha^3 + \alpha + 1 \quad (51)$$

which proves that the largest eigenvalue of  $T_{2,3,7}$  is

$$\lambda_1 := \alpha_1^{1/2} + \alpha_1^{-1/2} = 2.0065936183460167326505159176\dots \quad (52)$$

Thanks to help from Gert Almkvist, we proved that this generalizes to the relation

$$\lambda_m = \alpha_m^{1/2} + \alpha_m^{-1/2} \quad (53)$$

between the largest eigenvalue of the tree graph  $T_{2,3,6+m}$  and the Salem number obtained from (37), at any  $m > 0$ .

Thus this family of graphs generates a monotonically increasing family of Salem numbers, starting at the smallest yet known and tending to the provably smallest PV number (49). It is proven [11] that (52) is the smallest value,  $\lambda_{\max} > 2$ , for the largest eigenvalue of any graph. Sadly, this does not prove that Lehmer’s  $\alpha_1$  is the smallest Salem number.

Since Lehmer’s number field generates the new record,  $n = 17$ , for the order of a polylogarithmic ladder, we thought it interesting to examine ladders based on further members of the family of graphs  $T_{2,3,6+m}$ . By way of benchmark, we recall that the square of the golden ratio generates ladder (22), with order  $n = 9$  and index  $k = 12$ . The smallest PV number (49) also reaches  $n = 9$ , with index  $k = 42$  [15]. In [7], order  $n = 11$  was attained, from cyclotomic relations that also led to algorithms for finding the ten millionth hexadecimal digits of  $\zeta(3)$  and  $\zeta(5)$ , without computing previous digits. To our knowledge,  $n = 11$  had been bettered, heretofore, only in the Lehmer number field.

We were intrigued to know the maximum orders achievable in bases derived from trees  $T_{2,3,6+m}$  with  $m > 1$ . Accordingly, we studied the cyclotomic relations of (37), with

$m < 6$ , and found  $N_m(n)$  ladders, for base  $\alpha_m$  at order  $n > 1$ , with

$$N_1(n) = 77 - \lfloor 9n/2 \rfloor \quad (n \leq 17) \quad (54)$$

$$N_2(n) = 54 - \lfloor 9n/2 \rfloor \quad (n \leq 11) \quad (55)$$

$$N_3(n) = 49 - \lfloor 9n/2 \rfloor \quad (n \leq 10) \quad (56)$$

$$N_4(n) = 44 - \lfloor 7n/2 \rfloor \quad (n \leq 12) \quad (57)$$

$$N_5(n) = 43 - \lfloor 9n/2 \rfloor \quad (n \leq 9) \quad (58)$$

The number of cyclotomic relations decreases monotonically, as the Mahler measure increases from  $\alpha_1$  to  $\alpha_5$ . However  $\alpha_4$  carries the advantage of having the lowest degree,  $d = 8$ , among Salem numbers from trees  $T_{2,3,6+m}$ . The explanation is that the characteristic polynomial of the adjacency matrix of tree  $T_{2,3,10}$  is

$$P_{2,3,10}^T(\lambda) = (\lambda^4 - 4\lambda^2 + 2)P_{2,4,5}^T(\lambda) \quad (59)$$

with the same largest eigenvalue as for tree  $T_{2,4,5}$ , which is the first of a family of trees  $T_{2,4,4+m}$  whose largest eigenvalues are  $\beta_m^{1/2} + \beta_m^{-1/2}$ , where the Salem number  $\beta_m$  solves

$$x^{2+m} = \frac{R(1/x)}{R(x)} \quad (60)$$

with  $m > 0$  and

$$R(x) := x^3 - x^2 - 1 \quad (61)$$

These Salem numbers increase monotonically, from  $\beta_1 = \alpha_4$ , to

$$\begin{aligned} \beta_\infty &= \frac{1}{\left(\frac{\sqrt{31/27}+1}{2}\right)^{1/3} - \left(\frac{\sqrt{31/27}-1}{2}\right)^{1/3}} \\ &= 1.4655712318767680266567312252 \dots \end{aligned} \quad (62)$$

which is the PV number obtained from (61). As seen from (57), the first of these Salem numbers gives valid ladders up to  $n = 12$ . These entail 62 indices that divide elements of

$$\begin{aligned} \mathcal{T} := \{ &23, 24, 33, 40, 45, 50, 54, 55, 56, 60, 62, \\ &64, 68, 75, 78, 84, 88, 102, 105, 114, 140, 252 \} \end{aligned} \quad (63)$$

Finding two valid ladders at order  $n = 12$ , we obtained no definitive answer regarding the appearance of 691, the numerator of  $\zeta(12)/\pi^{12}$ .

This study of graphical number fields led to the observation that the Salem number

$$\alpha_{\text{not}} = 1.216391661138265091626806311199463327722253606570 \dots \quad (64)$$

that solves

$$\alpha^{10} - \alpha^6 - \alpha^5 - \alpha^4 + 1 = 0 \quad (65)$$

is rather special. It is the sole Salem number  $\alpha < 1.3$  with degree  $d < 12$  for which  $\alpha^{1/2} + \alpha^{-1/2}$  is *not* the largest eigenvalue of the adjacency matrix of a graph. We note that

Gary Ray [17] identified (65) as a potentially fruitful source of polylogarithm identities, along with (39,42).

Accordingly, we sought its cyclotomic relations and hence its polylogarithmic ladders, finding  $59 - \lfloor 9n/2 \rfloor$  valid ladders at orders  $n = 2 \dots 13$ . At  $n = 13$ , we found an integer relation of the form

$$u \zeta(13) = \sum_{j=0}^6 v_j \pi^{2j} (\log \alpha_{\text{not}})^{13-2j} + \sum_{k \in D(\mathcal{U})} w_k \text{Li}_{13}(\alpha_{\text{not}}^{-k}) \quad (66)$$

with integers (see footnote 1)  $u$ ,  $v_j$  and  $w_k$ , where  $k$  runs over 86 divisors of elements of

$$\mathcal{U} := \{32, 38, 40, 43, 48, 50, 54, 56, 57, 58, 60, 72, 75, 84, 88, 90, \\ 92, 110, 118, 124, 126, 132, 136, 156, 170, 204, 210, 234, 294\} \quad (67)$$

The coefficient of  $\zeta(13)$  was factorized as follows

$$u = 2^3 \times 3^5 \times 5^2 \times 7^2 \times 11 \times 13 \times 19 \times 43 \times 107 \times 1789 \times 3413 \\ \times 350162215794091 \times 31692786317928349 \times P_{94} \quad (68)$$

where the 94-digit factor

$$P_{94} := 80131278764880863222650485358197334145287799 \\ 30835715165005435238201238560738060500528870828871 \quad (69)$$

is (very) probably prime.

We believe that (66), with index  $k = 294$ , is the unique valid ladder of order  $n = 13$  in the number field (65). In our investigations, the only number field that yielded orders  $n > 13$  was Lehmer's. Thanks to the index  $k = 630$  in (1), the latter attains order  $n = 17$ . We expect this record to abide.

## Acknowledgements

DJB thanks Gert Almkvist, for an enjoyable stay in Lund, where the relation to graph theory emerged; Richard Crandall, for advice on factorization; Petr Lisonek, for discussions at Simon Fraser University, from which the idea of using PSLQ on the Lehmer number field originated; Chris Wigglesworth for expert management of a cluster of Dec-Alpha machines at the Open University; and Don Zagier, for discussions in Bonn and Vienna, which helpfully emphasized the reliability of (21).

**Table 1:** The 115 integers  $a_k$  in (31) yield (36) and have 11 digits less than those in Appendix A of [8].

$k$	$a_k$
1	657517430619563136979927560311907031673621211923757556039680
2	359886000860172792447593380451304667509342919316395916742210
3	-141684894142517938914234443949080358856724456088731117486080
4	196781701168242866806602275650062000534697577654107935879825
5	745935348767332443069712397147123742304948147537862045466624
6	405488816708650056442253795413843043306948102060187723235830
7	-416406976403628958702584514595218236213894104883105761525760
8	-547505077081618409188348198016100365604103632542087633890050
9	-168975342608695257899656804048229345162975246342759199539200
10	-230249343614447384417237255880683163036311216601594850723110
11	-686361499027198737843324163158516540202013109745360933355520
12	-555211669354532402241841641332915138353295668921519456494790
13	192815301162191732499187837598868261035304857047105259765760
14	679734532959241358849438888514522682758692394636208781152890
15	542262843988600709919208184100477463027334034487684580835328
16	424235600610741544849445172143286300309734125137633061204035
17	523715028246745518068354598319413022191932310750064994549760
18	391765228725624091642051568538398618159246632370196009838220
19	-198765352117361763621609618366336349702431673161386991943680
20	-681771795414951811771620988305943452975785965232369963176182
21	-711962908052232232505015799802945837905471681104673666826240
22	-474711713652489086239653502137393573002849538293395396856580
23	-305563488793803341514102820998121597667241109957343639961600
24	-72481617150888165437226250731733584917169373028647243993490
25	277441845925552929689261105309806909812357891559963458994176
26	560813990488672759149774541727869507575569280403768300321140
27	529382955893422715273603248526440343418480284327032762204160
28	320081042366233059912738009855303427011191818027491322896270
29	83494538290613239874707297912762677581023131627678971658240
30	-98012487241465589885244822793749074739104108244040188504014
31	-259850303587986534656435580762491706946152146552951829626880
32	-312381473762254154637700526528599595535218857011814459284405
33	-246779089811622911261098199061852288809784802679547056947200
34	-101370164345933351431594273523846287706927174737460926584690
35	10117727926311381644405806386440429351025303665206665019392
36	92813012460581905456644049699805057100444523762661712331760
37	126324305118118269024517948173064327264644326236420977131520
38	115549700605836091570299617434612530310676419462990155960060
39	55524385597088576211029994800590286128713854655226225950720

40	8681384820081409297328583655855503616180543689183078659790
41	−19635305068579015455281545202962471423318427323980082053120
42	−25853290618480167202972487269611444701702175335712826764200
43	−31825441142023707248687536800399542265330919953456915087360
44	−21238990403232997150237193584099678135190886191312613084840
45	−5941171482587742832139996778309010890215227722504605532160
46	6867449991711733361905911701040369140850427873891452167820
47	2683941033568052348250476195016210444676454909493450178560
48	750276335951027139956797045951168533453325950288012793755
49	902953609381753090186332205561649133432166531686386565120
50	2845791617893774854256316723847280908911507508173997476472
52	−211518744564361609748949526157272452992328540189820187580
54	1586305520716919558891270499363536151171957733664553920190
55	317399573539556012199105353632753466414027589073604640768
56	−411955065839184813654206085597826201289284276111276432380
57	−633034566415505822568457398619791655535556228581668945920
59	−422235192545960711449176250527328955777038182764768133120
60	−100775809060677875668353632672084730591567385836002358340
62	103161260941635325589940465107263862028544860193511288430
63	−71239824525936132532239108327787334379823490760552611840
64	−758961502664172364046541735177189723413489520451845090
65	−31614329666390498308045970054858836918855503060815904768
66	82590308807941016435157163298430216384039435573691923590
68	−6111893964221453989555021260745192324274991457297917375
69	36723207357049059339848864763934042997356103557409013760
70	56587501615035807034718738004427311071764030581924717666
72	6246571495623829969272301654360557650197045971497886770
74	4662327569727635575068647456890293743838455770712101950
75	14887216546252872098059151093987262212120215877194153984
76	4933420845476441755128409798981902904608462970609785585
77	−3718375935394449285719242013677644130212998760206499840
78	797543027544694301981107935492458580524804545433769960
80	−756655048589380952520516473920663602783235627959554055
82	599221956438568586892136999602126203104200052611696840
84	1711070904277206791256558392136135387624754115179524010
86	971235386414297706563950708020005562296475828657742770
90	712743581327703411348920012655781994420044560878114988
91	581533337739120206692138569579040767870857035407360000

92	281171208042281370705030709443218746311661931924480940
93	−123474844527238243730367824366624184577805521172234240
96	45358670699766680263040092382358986091316360899268450
98	−27555957317558382879221563890431186933354691518749590
102	−13957477138823786670980509990907214192819210555392000
105	−51592349989081948844625156379637404692389576604581888
106	−49757317433194357135489925493889989029913561751388160
108	4104159613575523414703444177171856613076465917080545
110	436503492818404043948452074062000221710038770175550
118	12885595475645773664830818192362333855500432823631840
120	−4458271535866469116508257194072443804522804186654722
123	684209085353296089217162854388925631175894697902080
124	−2568170165780877908485719228826518196430780458722240
126	−1238638691691188975878629542803876608020141601562500
130	1661804350411188506429822540247968366095406030963502
132	−1633602118796242535519471886453758168320481063983580
135	1847684862191853167535965349249779010200990442520576
138	−1120703349519319437861598656125916839518924058758820
143	174515941261835218319905200778076418833749415362560
144	−317934100357607293764664092299747258866505243251140
154	113475828106520058768287414968189823309722862555130
160	−10275921042899706253486332862758642766530474426501
165	−23116946816383260126447835448067026267507493044224
175	−32347514260620927078585304389860102844649182658560
180	−4479106615252570847623887867187960984130837217280
182	−17746989066745611776493486620454124996058869488750
186	3768153214332221793529291515094732195367600133430
195	−2675778372359720766614632517822289310362850099200
204	425948399011956380339981383999853948755469072125
210	−291194657074029253989297333815629327367478738414
212	1518472822057933262191465011410216950375780082745
240	76199283284999932540837893030287171626166802389
246	−20880404216104006628941737499662037084225302060
270	−56386867132319737778807536293023041082793897782
286	−5325803871515967355954138207338757898979169170
315	−664989084046735447844305843955260271683239936
360	136691486061174647449459468603148223392664710
630	20293856324668440180795466429298714345802



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